EMERGENCE OF UNIQUE PARAMETRIC DEPENDENCE IN AN EPIDEMIC MODEL.

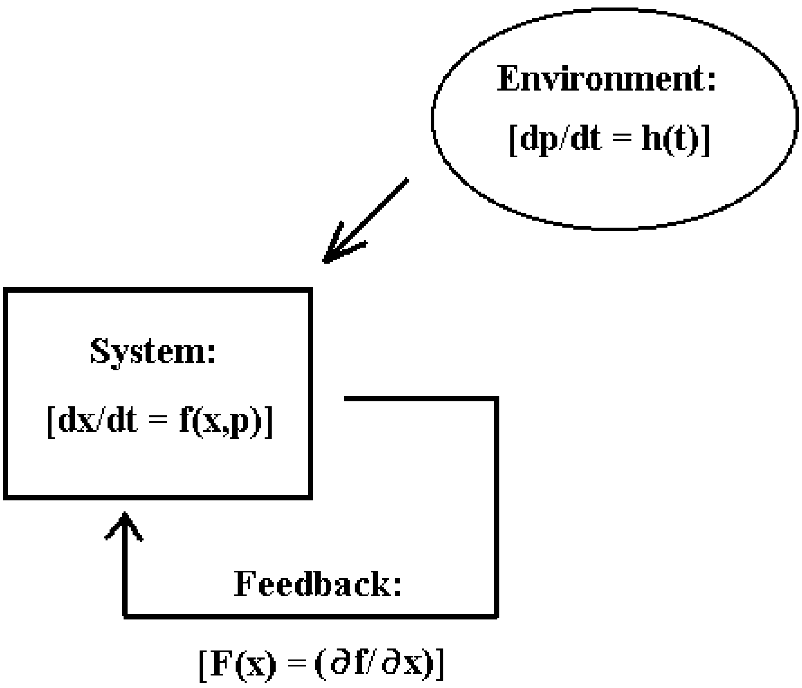
W. M. Schaffer

T. V. Bronnikova

04/05/05

Dynamical Systems.

- **Mathematical objects** used to model the *time evolution* of real-world systems.



- Regardless of application, we inevitably distinguish **system** from **environment**.

- Environment may or may not change with time, but in either case is **unaffected** by state of the system.

Dynamical Systems (continued).

- Time evolution of a dynamical system determined **jointly** by the environment and **internal feedback**.[1]

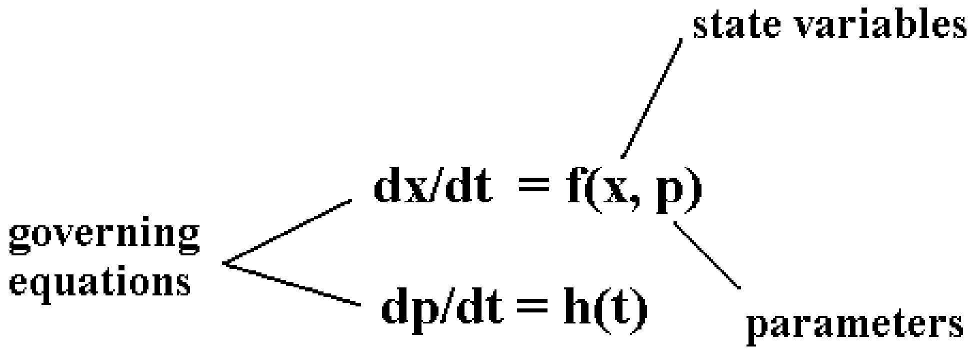
- Dynamical systems can be categorized in many ways:
  1. **Continuous** *vs.* **discrete**.
  2. **Autonomous** (constant parameters) *vs.* **non-autonomous** (time-varying parameters).[2]
  3. **Finite** dimensional *vs.* **infinite** dimensional.
  4. **Conservative** *vs.* **dissipative**.
  5. **Deterministic** *vs.* **stochastic**.

---

1. Distinguishing system from environment: straightforward **in principle**; often difficult **in practice**.

2. Non-autonomous systems are equivalent to **higher dimensional** autonomous systems

Representative Dynamical System.



- **Governing equations** “determine” the motion.

- **State variables** are the quantities that “evolve.” Collectively, they specify the **state** of the system.

- **Parameters** encapsulate the environment.

- Parameter values can change with time, but their time evolution is **unaffected** by the state of the system.

Studying Dynamical Systems.

- **Two Complementary Approaches.**

  1. Study the time evolution of the **system state**, (x(t), y(t) , …) for **fixed** parameter values.

  2. Study the birth / death of **attractors** and **other invariant sets** as one varies parameters.

- In the first case, one works in **phase space**; in the second, in **parameter space**.

- A good example of the way in which these approaches complement each other provided by the **complex logistic map**

$$Z_{i+1} = Z_i^{\;2} + C \quad , \qquad (1)$$

where Z and C are complex numbers.

Mandelbrot / Julia Sets.

- Governing equation can be rewritten as

$$X_{i+1} = F(X_i, Y_i, P, Q)$$

$$Y_{i+1} = G(X_i, Y_i, P, Q) \qquad (2)$$

- For every pair (P,Q) of parameter values, ∃ an **attractor** at (X,Y) = (-∞, -∞).

- For some (P,Q) ∃ also a *finite-valued attractor* (FVA) – a **fixed point** or a **cycle**.

- Each FVA has a **basin of attraction** (BA) in the X-Y plane. The boundary of the BA is called a **Julia set**, *J*.

- The **set of (P,Q) values** for which there **exists an FVA** is called *M*, so named by Benoit Mandelbrot. [3]

---

3. With (P,Q) on the boundary of *M*, *J* is a fractal.

Mandelbrot / Julia Sets (continued).

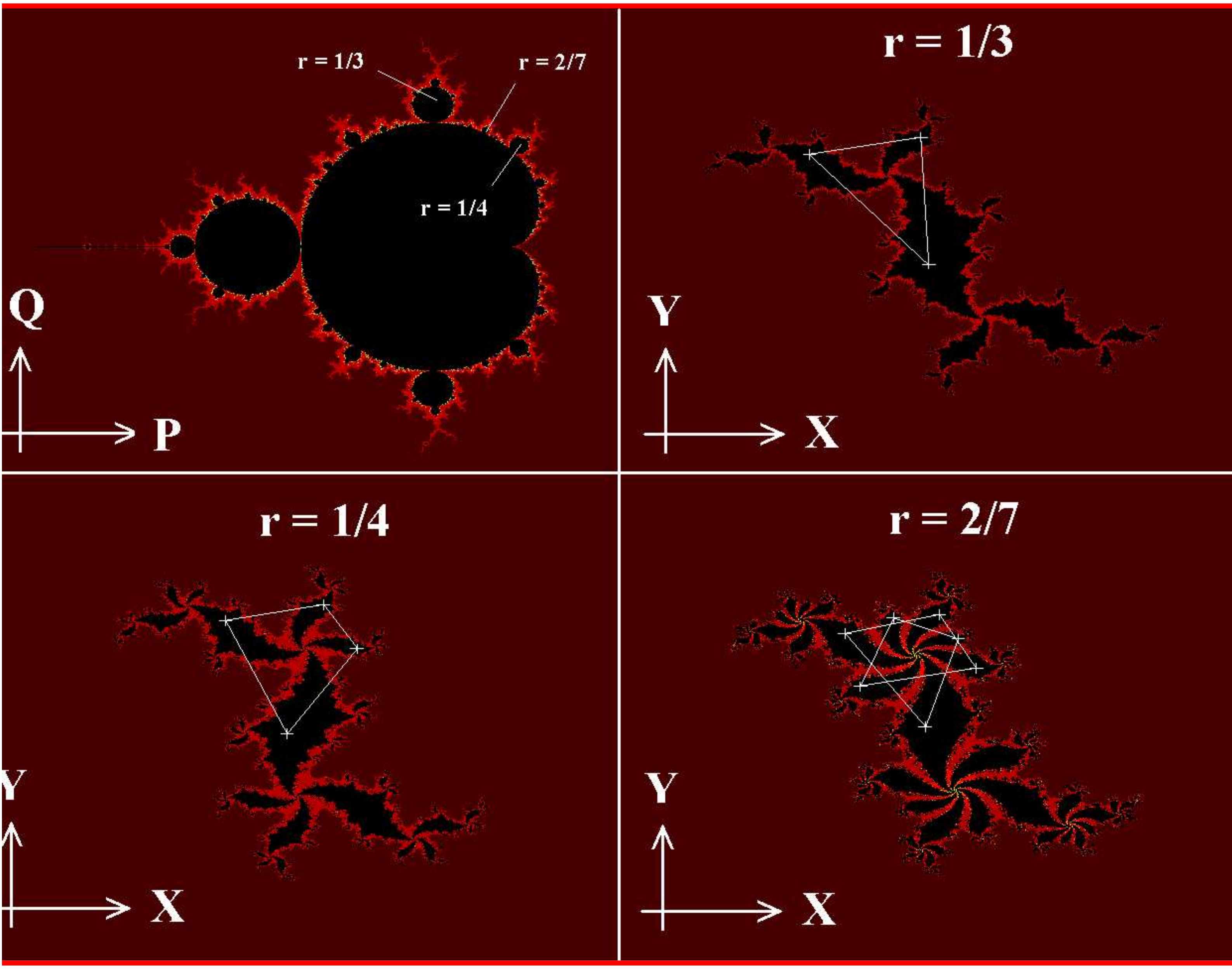
- Each FVA can be characterized by **its rotation number**, $\rho = n/m$, where
  1. m = number of points on the cycle;
  2. n = number of 2π rotations required to visit all of them.

- *M* consists of an infinite number of **connected regions** (the “balls”) corresponding to FVAs of **different** rotation numbers.

- The regions are ordered (as one moves along the margin of *M*) according to the **Farey Sequence** (see below).

- *Modulo* the existence of the attractor at infinity, Eqs (2) manifest **unique parametric dependence** (UPD) – for each (P,Q) there is **one and only** one FVA.

Mandelbrot / Julia Sets (continued).



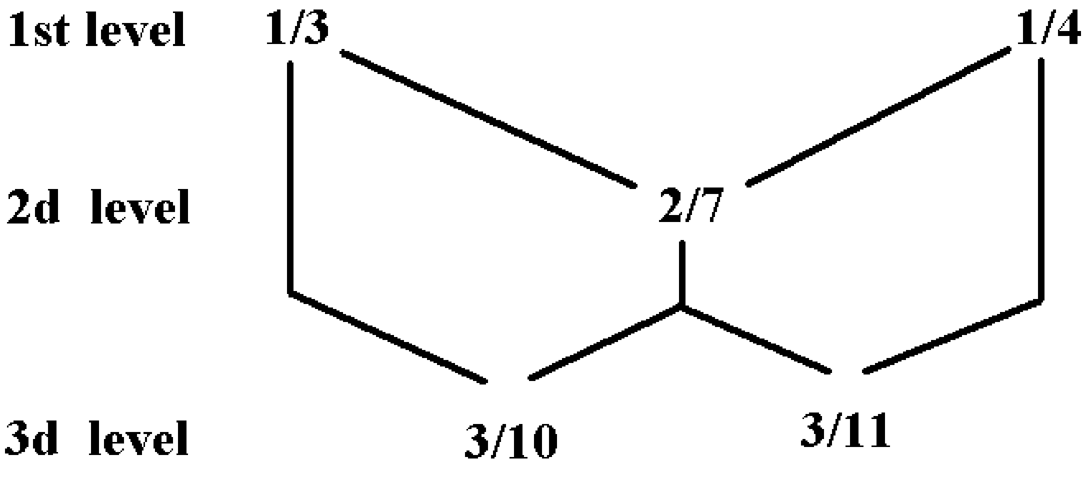
Note on Farey Sequences.

- A way of constructing the **real** numbers from the **rationals**.

- Successive **levels** of the **Farey tree** constructed via **Farey addition**, $\oplus$:

$$\frac{n}{m} \oplus \frac{p}{q} = \frac{n+p}{m+q} \qquad (3)$$

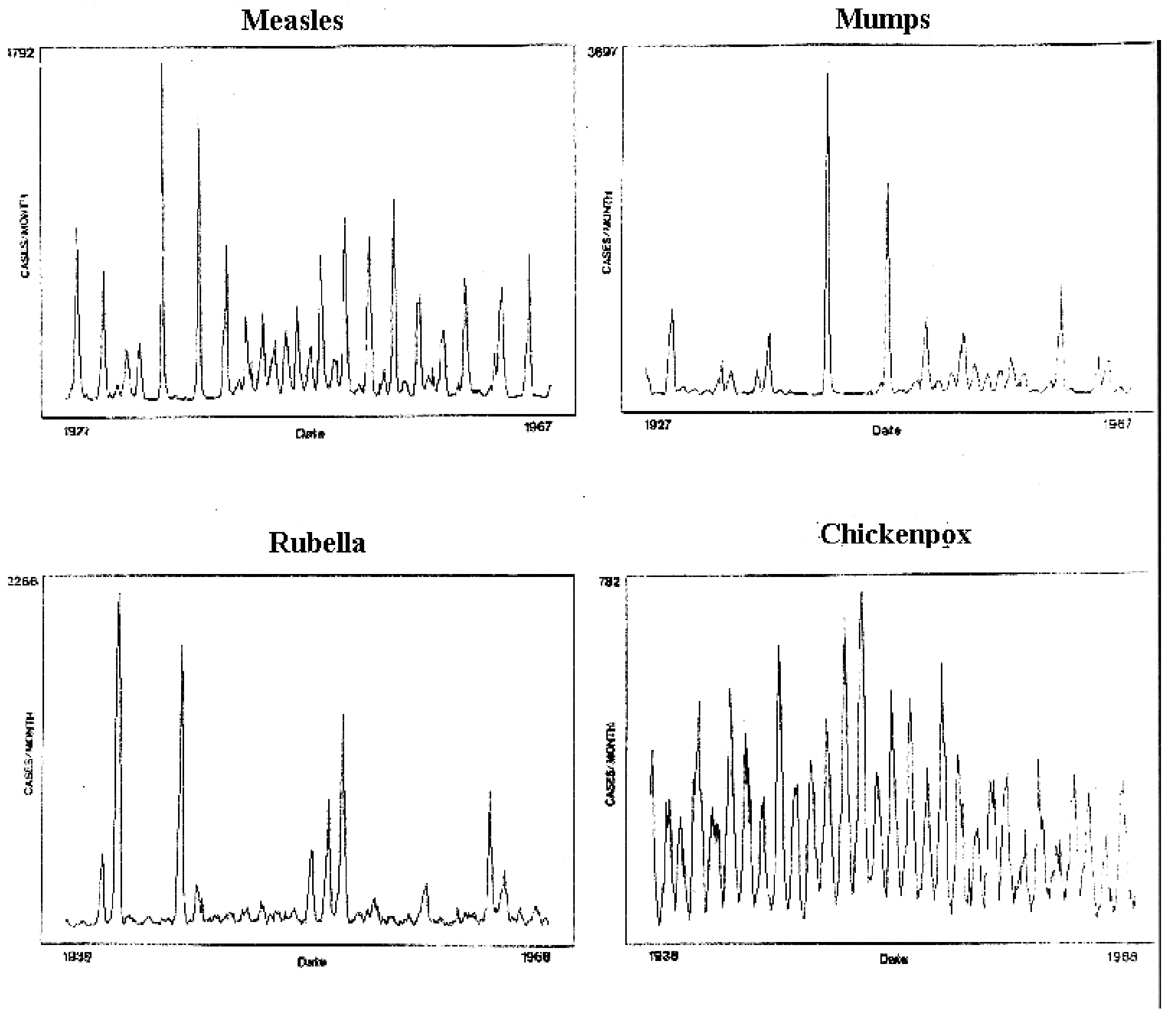
Farey Sequence Construction



Childhood Diseases.

- **Chickenpox**, **measles**, **mumps**, **rubella**.
- Diverse dynamics ranging from **annual** to **biennial** cycles to **putative** chaos**.**

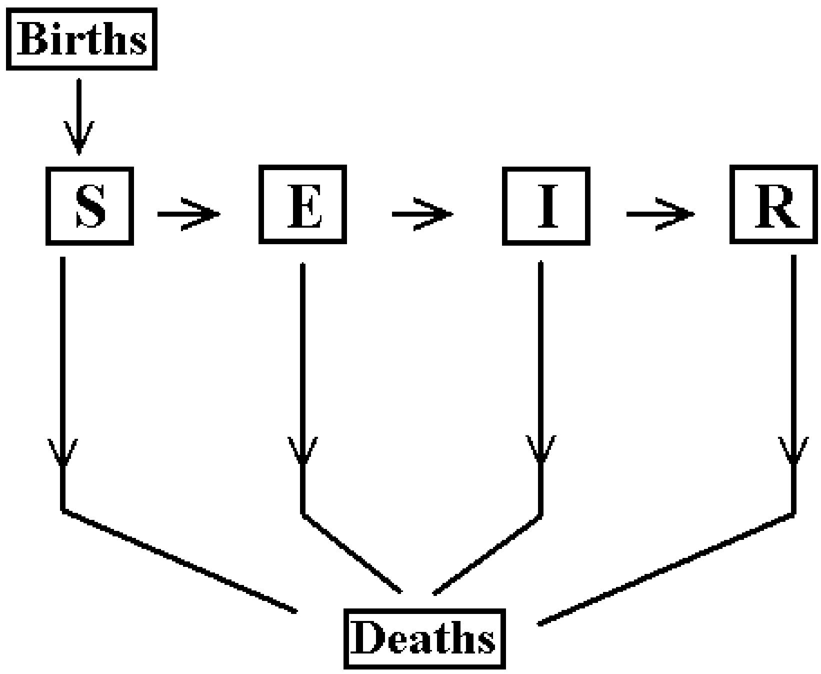
Copenhagen Notifications (1928-1967).



SEIR Equations.

- Appropriate for **microparasitic** infections.

- Categorize host individuals according to disease status:
  1. **Susceptible**;
  2. **Exposed** (but not yet infectious);
  3. **Infectious**;
  4. **Recovered** (and immune).



SEIR Equations (continued).

- **Typical Implementation:**
  1. **Constant** population size.
  2. **Quadratic** transmission ($\propto$ S$\times$I).
  3. **Linear** transitions from **E** to **I** & **I** to **R**.
  4. **Seasonal variations** in transmission – "school year effect" – modeled as simple trig function.

- **Thus,**

$$\begin{aligned} \dot{S} &= m(N - S) - B(t)SI \\ \dot{E} &= B(t)SI - (m + a)E \\ \dot{I} &= aE - (m + g)I \end{aligned} \tag{4a}$$

- **where**

$$B(t) = B_0(1 + \varepsilon_B \cos 2\pi t) \tag{4b}$$

SEIR Equations (continued).

- **Parameters** interpreted as follows:

N	host population size;
m^{-1}	host life expectancy;
a^{-1}	mean latency period;
g^{-1}	mean infectious period;
B_0	mean contact rate;
ε_B	seasonal variation in contact rate.

- Eqs (4a) often **non-dimensionalized** by substituting

$$s = S/N; \quad e = E/N; \quad i = I/N; \quad \beta_0 = B_0 N \qquad (4c)$$

- **Yields**

$$\dot{s} = m(1-s) - \beta(t)si; \qquad \dot{e} = \beta(t)si - (m+a)e$$
$$\dot{i} = ae - (m+g)i; \qquad \beta(t) = \beta_0(1+\varepsilon_B \cos 2\pi t) \qquad (4d)$$

Autonomous SEIR Dynamics.

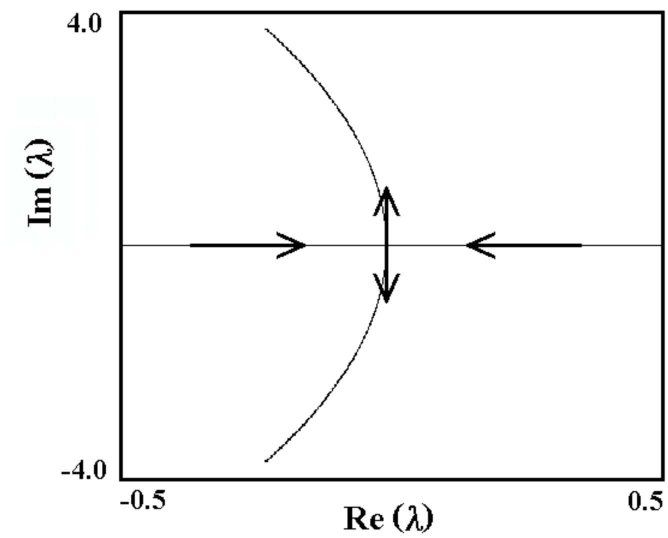
- **Two equilibria:**
  1. No disease state: (N, 0, 0).
  2. Endemic state: (S^*, E^*, I^*).

- **Transcritical bifurcation** at

$$R_0 = \frac{aB_0N}{(m+a)(m+g)} \approx \frac{B_0N}{g} = 1 \qquad (5)$$

stabilizes I^* as it **passes through** 0.

- Thereafter, two of the eigenvalues **complexify** ➔ damped oscillations.



Non-Autonomous Dynamics.

- **Wide range** of behaviors:
  1. **Annual cycles** and modulations thereof.
  2. **Multi-annual** cycles (1 peak per cycle).
  3. **Transient** and **asymptotic chaos**.

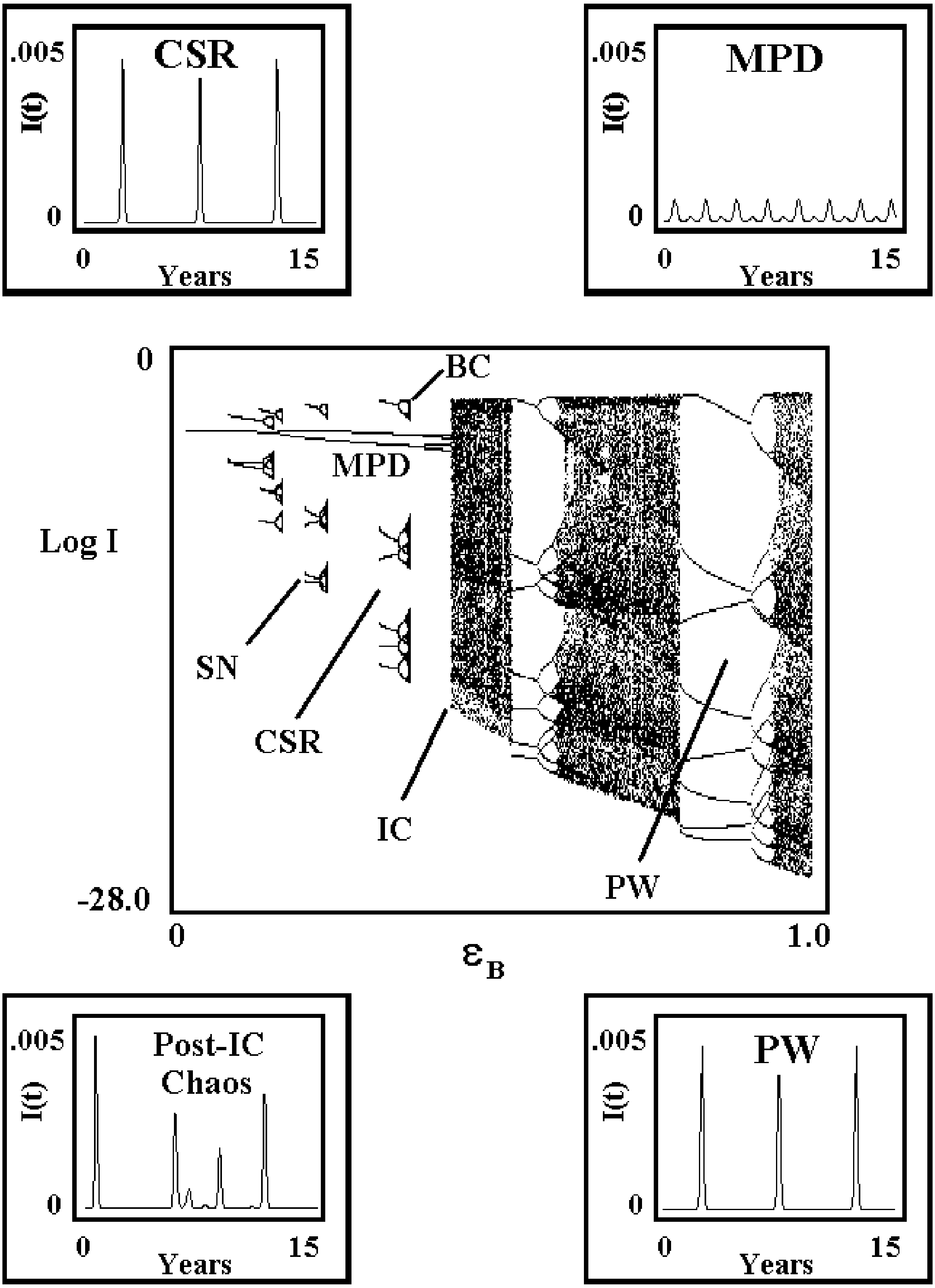
- Bifurcation diagrams (asymptotic dynamics[4] *vs*. B_0 or ε_B) often manifest:
  1. Main period-doubling (MPD) route.
  2. Coexisting subharmonic resonances (CSRs) before the IC.
  3. Interior crisis (IC) on MPD.
  4. Period windows (PW) in the main sequence *after* the IC.

- **Chaotic saddles** (not shown) also coexist with the MPD before the IC.

---

4. Trajectories sampled **strobiscopically**, *i.e.*, at intervals equal to period of forcing. This is a convenient way of discretizing continuous trajectories in presence of periodic forcing.

β_0 = 1500.


CSR
.005
I(t)
0
0
Years
15
MPD
.005
I(t)
0
0
Years
15
0
BC
MPD
Log I
SN
CSR
IC
PW
-28.0
0
1.0
ε_B
Post-IC
Chaos
.005
I(t)
0
0
Years
15
PW
.005
I(t)
0
0
Years
15

Non-Autonomous Dynamics.

- **MPD attractors**
  1. **Modulations** of the annual cycle.
  2. Manifest **period-doubling** to chaos.

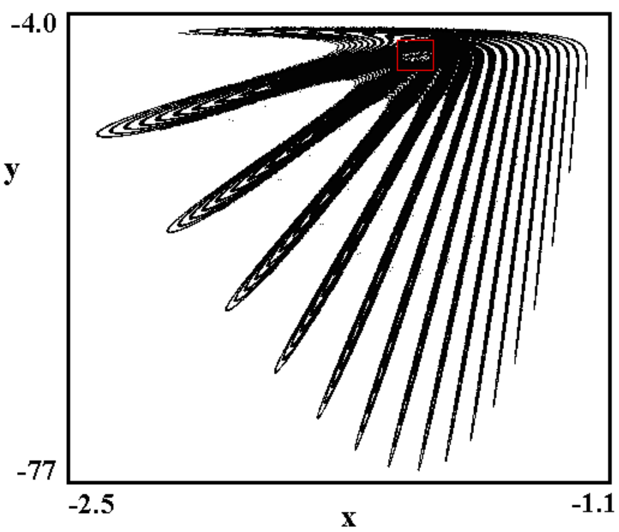
- **CSRs**
  1. Produced in pairs (one stable, one not) by **saddle-node bifurcations** (SN).
  2. Stable cycles undergo their own sets of **period-doublings**.
  3. CSR attractors destabilized **by boundary crises** (BC) whereat they collide with their basin boundaries.

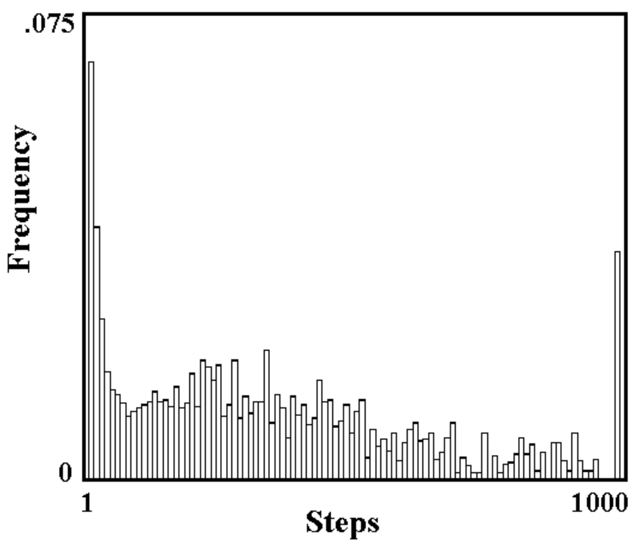
- **IC**
  1. Results from **collision** of MPD attractor with destabilized CSR.
  2. Produces **qualitative change** in MPD dynamics.

- **Chaotic saddles** (CS) also coexist with the MPD before the IC.

Transient Chaos.

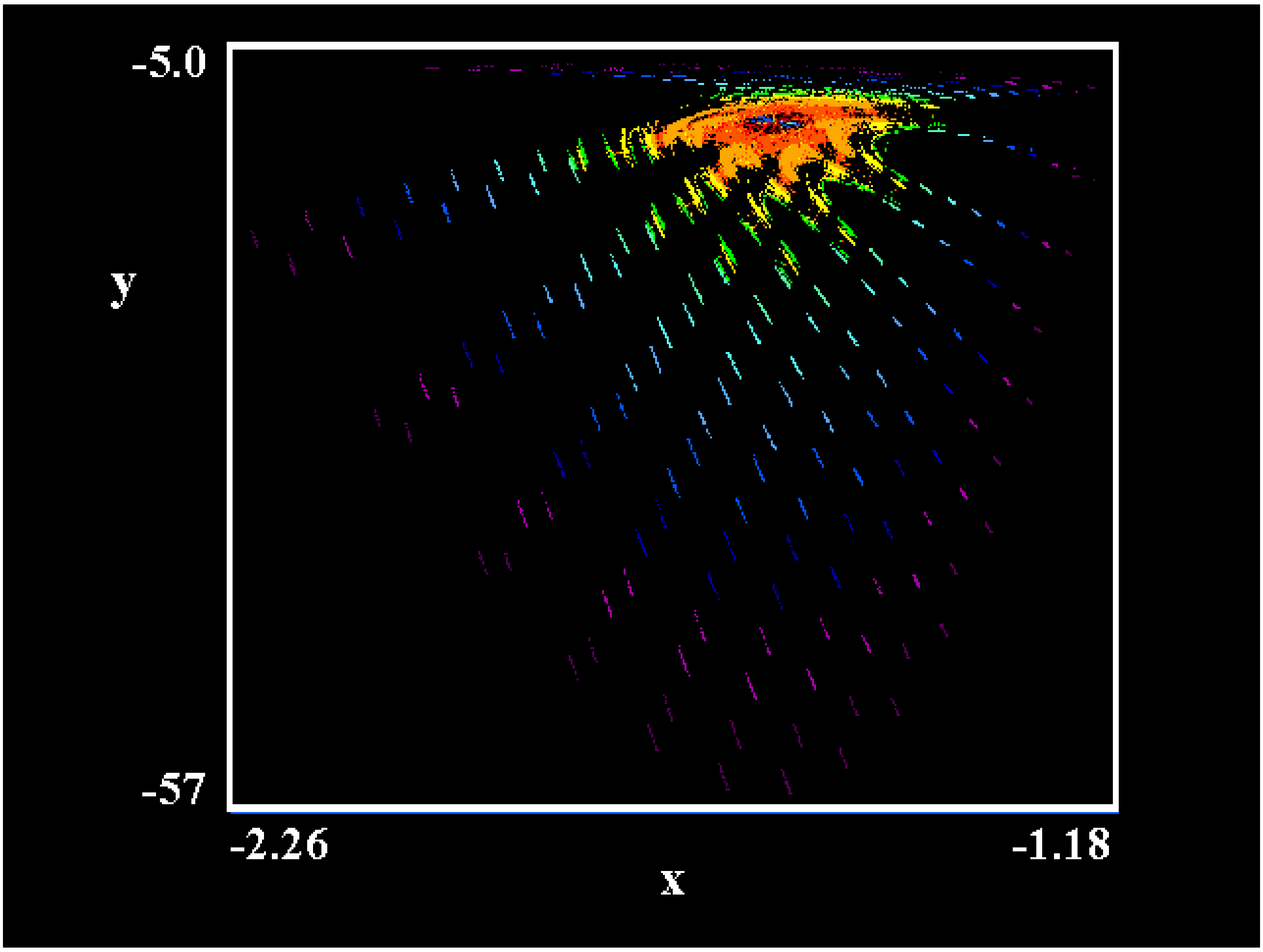
- Consequence of **chaotic saddles** that result from destabilization of CSRs via BCs.
- Even though all orbits **eventually** terminate on the MPD attractor, they can spend considerable time **shadowing** the chaotic saddle before settling down.
- **Organized** about “skeletons” of non-stable periodic points.





Transient Chaos (continued).

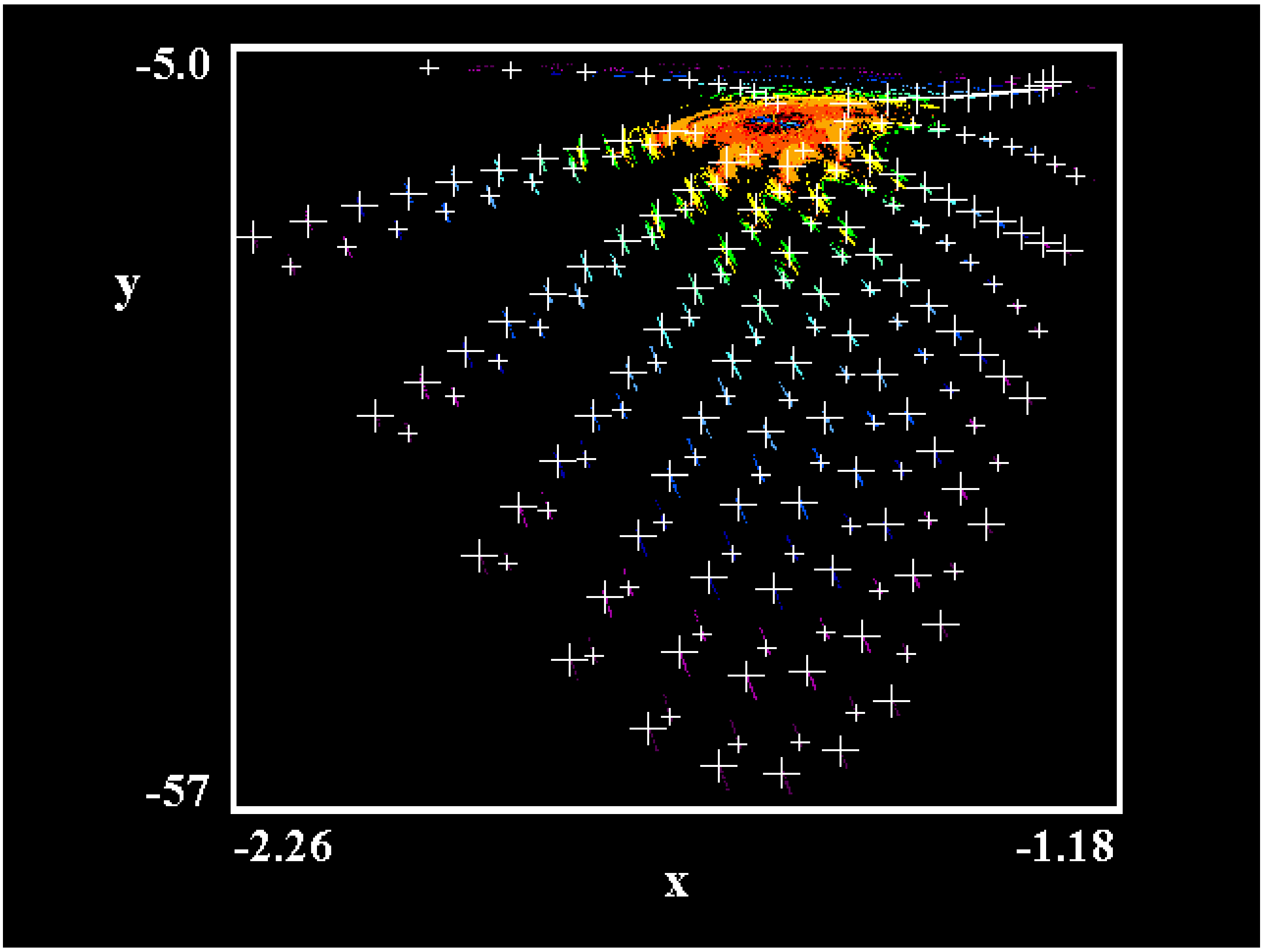
- Color-coding recurrent points on the **CS** allows one to visualize **a few** of the **periodic points** and their **stable manifolds.**[5]



5. Stable (unstable) manifolds are **nonlinear continuations** of the **linear spaces** spanned by the stable (unstable) eigenvectors of a fixed or periodic point.

Transient Chaos (continued).

- Color-coding recurrent points by **mean peak-to-peak interval** = 1/ρ. Subharmonic cycles with ρ = 1/14, … , 1/3 superposed. (B_0 = 500; ε_B = 0.25.)



Emergence of Parametric Uniqueness.

- With increasing B_0, the CSRs move to the **right** and are **incorporated** into the MPD.

- Incorporation **begins** when the BC that terminates a CSR bifurcation sequence **coincides** with the IC on the MPD.

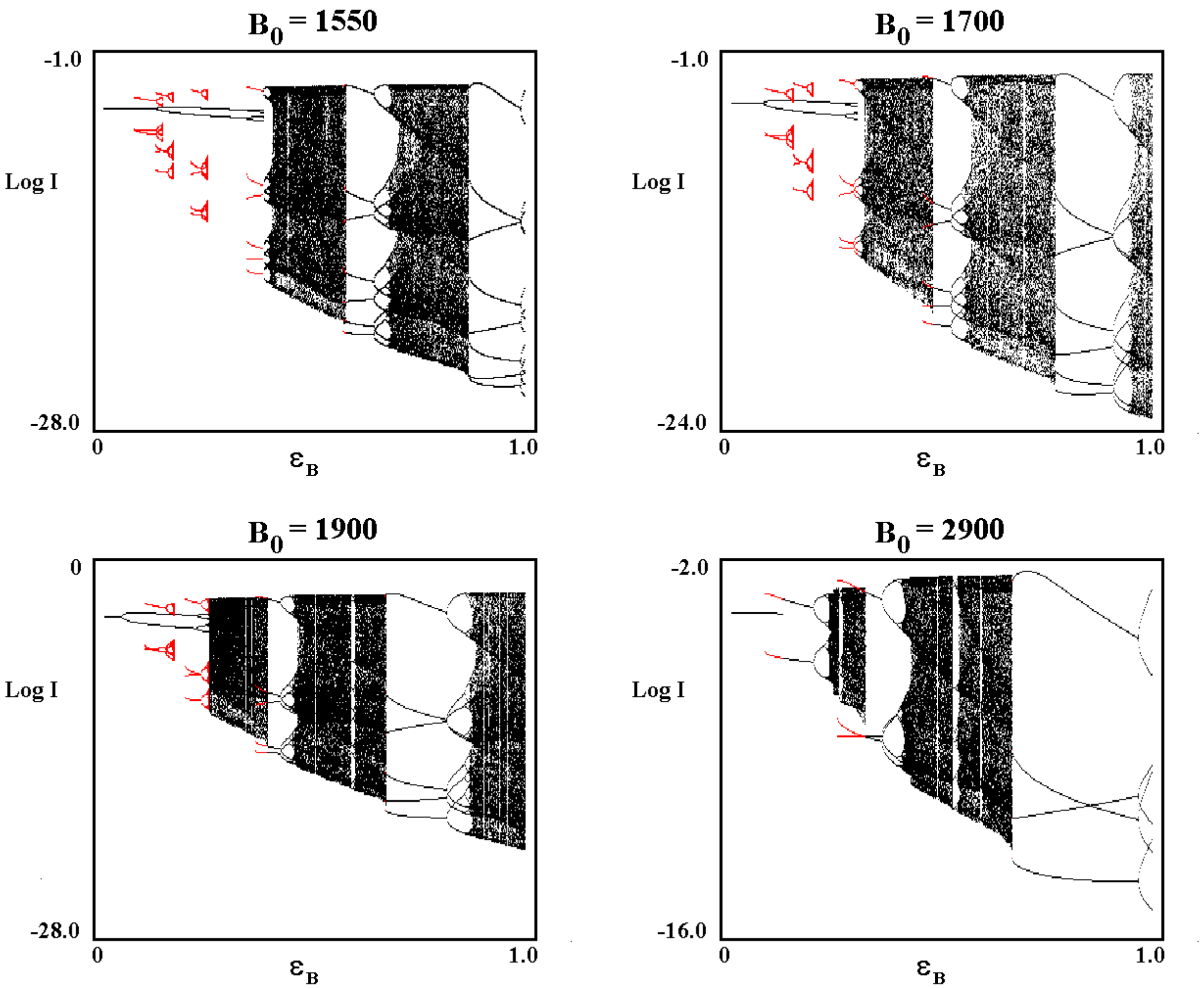
- Incorporation **concludes** when the SN that begets the CSR coincides with the IC.

- Between these two values of B_0, the CSR bifurcation sequence is **partly on** and **partly off** the MPD.

Parametric Uniqueness (continued).

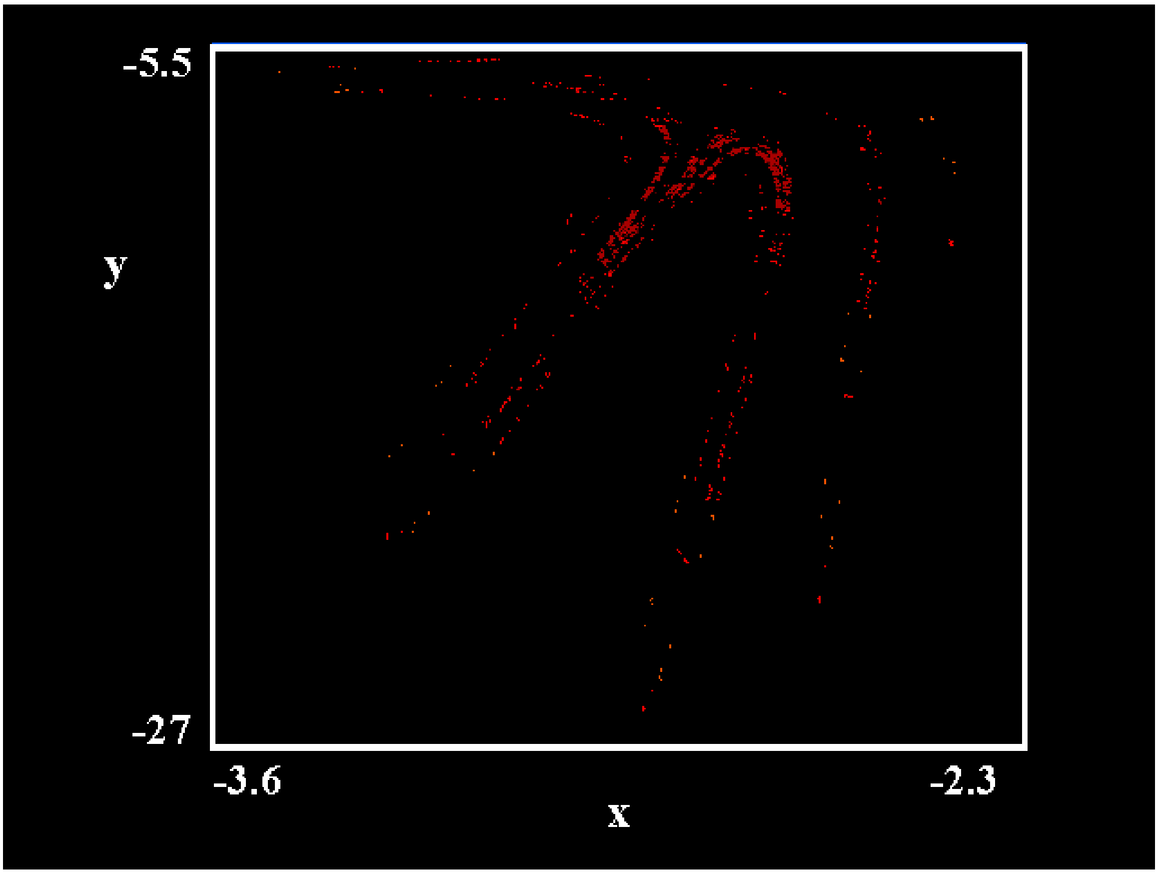
Incorporation of CSR Bifurcation Sequences into MPD Sequence.

(Coexisting attractors shown in red)



Chaotic Saddle Simplification.

- Emerging parametric uniqueness **simplifies** transient chaos – CS **loses** frequencies.



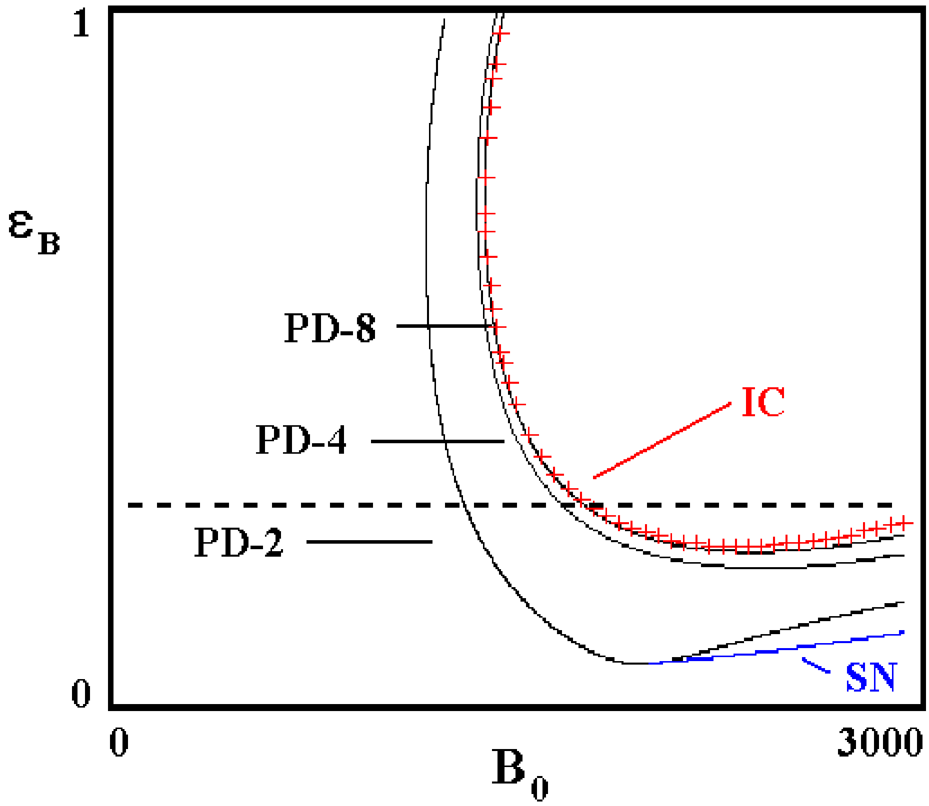
Chaotic saddle color-coded by mean peak-to-peak period (dark red = 1; dark purple, 14) for B_0 = 1800; ε_B = 0.27. Compare with previous pictures.

Two Parameter Analysis.

- Continue period-doubling and saddle-node bifurcations **numerically** in B_0-ε_B plane.

- Compute loci of interior and boundary crises by **brute force** inspection.

MPD Bifurcations.

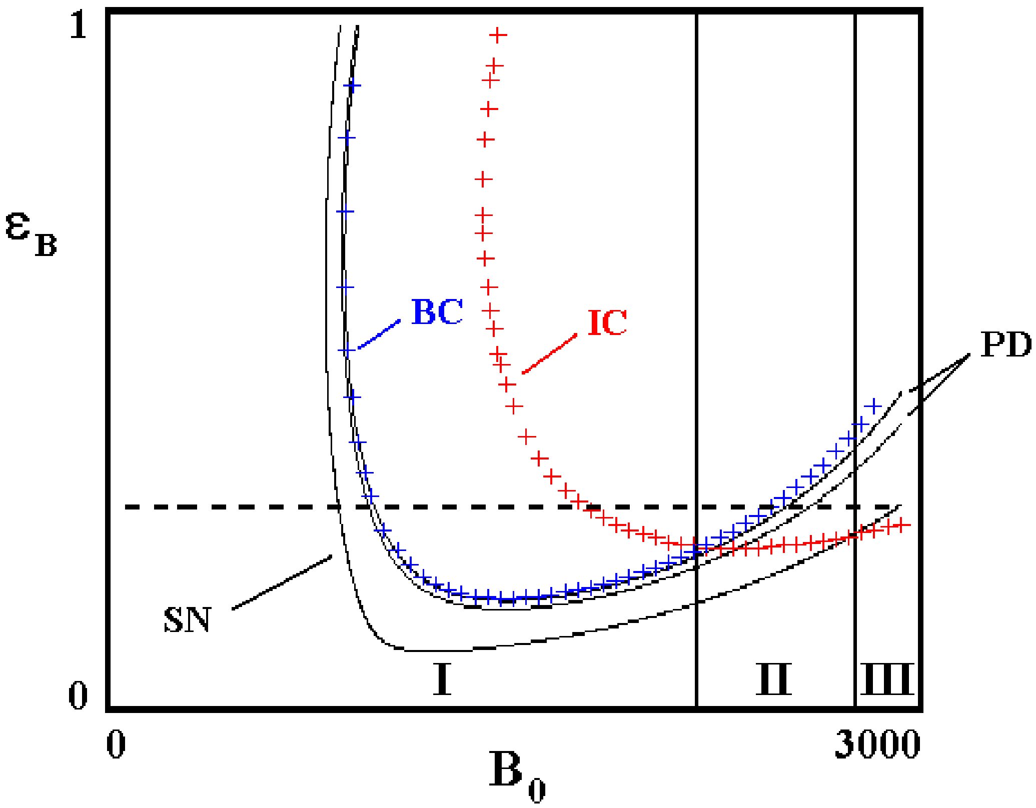


Two Parameter Analysis (continued).

- SN curves delimit **resonance horns** within which all invariant sets have same ρ.

- Superposing BC / IC curves shows **initiation / conclusion** of CSR incorporation.

ρ = 1/3 Resonance Horn.



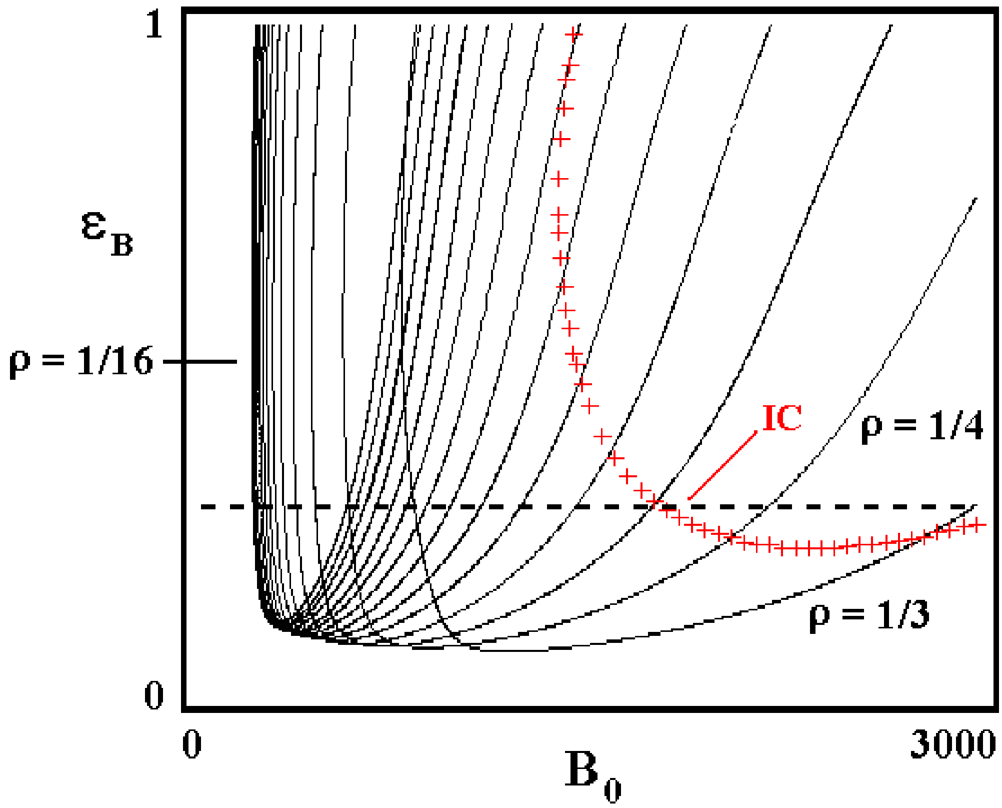
Two Parameter Analysis (continued).

- SN-IC and BC-IC intersections permit **division** of parameter plane into 3 regions:
  I. CSR entirely **off** the MPD .
  II. CSR entirely **on** the MPD.
  III. CSR partly on and partly off the MPD.

- Intersections are effectively **codimension 2** bifurcations in the sense that they can be continued by varying B_0, ε_B and a third parameter – *a* , *g* or *m*.

- Displaying multiple SN curves and the IC curve indicates
  1. **Accumulation** of (an infinite number?) of CSRs at a critical value of B_0 (left).
  2. **Sorting out** of 1st level Farey sequence CSRs by rotation number in response to increasing B_0.
  3. **Origin** of and post-IC **sequence** of periodic windows within the MPD.

Two Parameter Analysis (continued).



- An apparently **infinite** number of resonance horns accumulate at $B_0 \approx 200$.

- With increasing B_0, one **exits** high period resonance horns, and **enters** (and then exits) horns of lower period.

- Intersections of SN and IC curves mark incorporation of CSRs into MPD.

Conclusions.

- Emergence of UPD in response to increasing values of B_0, reflects increasing **dissipation** ("friction") as indicated by increasing negativity of the divergence of the vector field, *i.e.*,

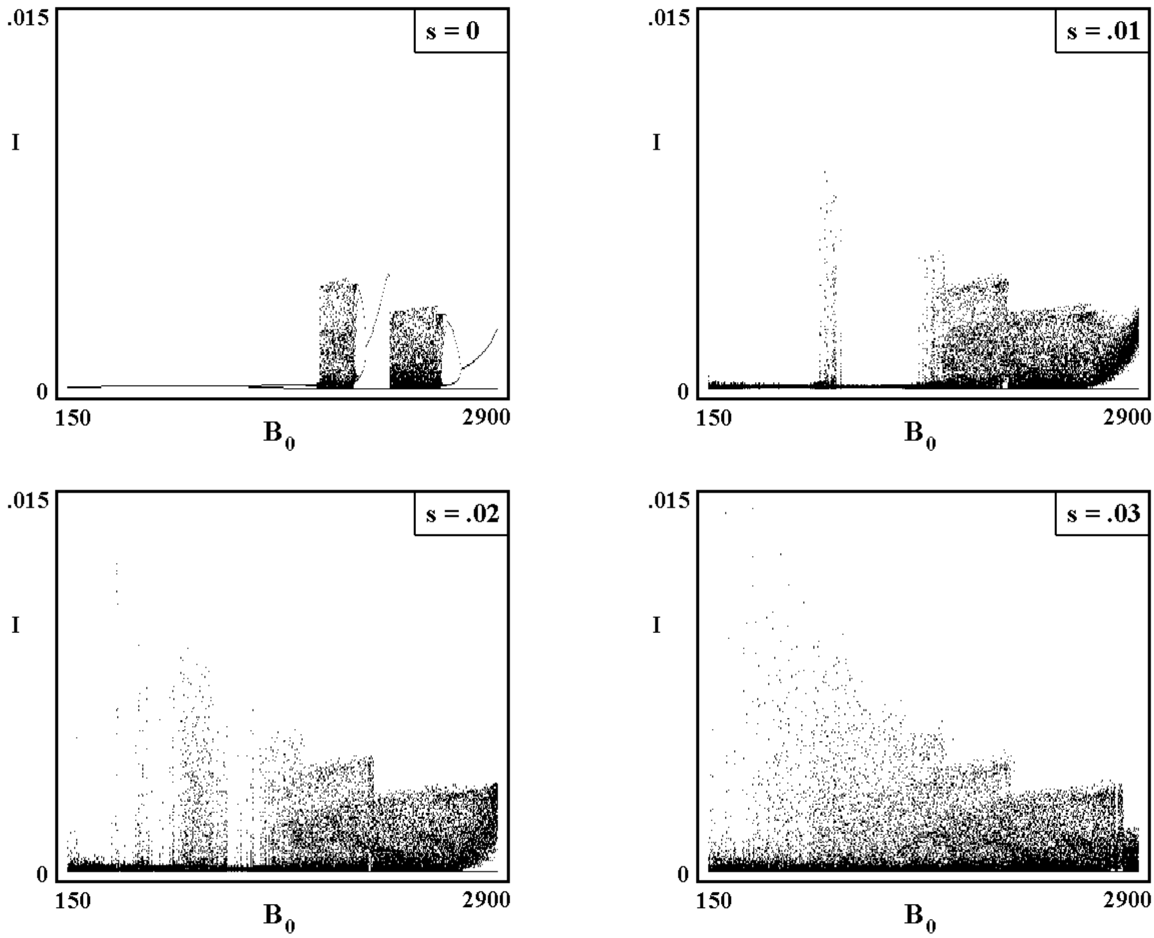
$$\frac{d}{dB_0}(div\ F) = -I \tag{6}$$

- Similar result previously hold for seasonal predator-prey models.

- ∃ a **trade-off** between the complexity of **transient** and **asymptotic** dynamics – *i.e.*, with increasing dissipation, CSRs move off the CS and into the MPD.

- ➔ Noise-mediated reversal (NMR) of parametric dependency – *i.e.*, qualitatively **different** bifurcation diagrams obtain with and without noise.

Noise Mediated Reversal of Parametric Dependency.



Bifurcation diagrams corresponding to dashed line in previous figures computed in the presence of increasing levels of multiplicative noise: *S*, *E*, *I*, replaced by *S(1+u)*, *etc.*, where *u* is uniformly distributed on [-s, s] as shown. In the absence of noise, dynamical complexity increases with B_0. With *s* = .03, the pattern is reversed.

Conclusions (concluded).

- Mumps, rubella and chickenpox have values of B_0 roughly **half** that of measles.
  1. **Accounts** for **mix** of annual and multi-annual cycles in **mumps** and **rubella**.
  2. Does **not** account for absence of multi-annual cycling in **chickenpox**.